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## THEORETICAL AND EXPERIMENTAL DETERMINATION OF DAMPING CONSTANTS OF ONE-TO THREE- DIMENSIONAL VIBRATING SYSTEMS



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**THEORETICAL AND EXPERIMENTAL DETERMINATION  
OF DAMPING CONSTANTS OF ONE- TO THREE-  
DIMENSIONAL VIBRATING SYSTEMS**

by

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## ABSTRACT

Formulas are deduced for vibrating systems of one, two, and three dimensions. Undamped and damped free vibrations and harmonic forced vibrations are treated. Methods are proposed for calculating the damping constants from test observations.

## I. INTRODUCTION

In developing formulas for vibration and possible flutter of structures such as rudders,<sup>1, 2</sup> it may be necessary to include damping forces. Since these forces are not easy to calculate, methods of determining them from test observations may be needed.<sup>3\*</sup> The basic theory for two- and three-dimensional cases will be considered and feasible methods of observation will be sought. First, however, formulas for the one-dimensional system will be written to assist in treating the main problem. For convenience of reference, a summary of the results is given in Table 1; see pages 30 and 31.

## II. ONE-DIMENSIONAL VIBRATIONS

Assume as the equation of motion

$$m\ddot{x} + c\dot{x} + kx = P(t) \quad [1]$$

in which  $m$ ,  $c$ , and  $k$  are positive constants,  $\dot{x} = dx/dt$ , and  $P(t)$  denotes an applied force varying with the time  $t$ .

<sup>1</sup> References are listed on page 33.

\* In Reference 1 (see pages 78 and 83), certain damping terms were omitted from the flutter equations because methods for determining these terms from experiments were unknown to the authors at that time. These flutter equations including the damping terms originally omitted are of the same form as the equations given here for the three-dimensional case.

## 1. DAMPED FREE VIBRATIONS

If  $P = 0$ , the general solution of Equation [1] can be written (as is easily verified) as follows in terms of independently arbitrary amplitudes  $a$  and  $b$ :

If  $c^2 < 4mk$  (less than critical damping):  $x = e^{-\mu t} (a \cos \omega t + b \sin \omega t)$

where  $\mu = \frac{c}{2m}$  and  $\omega^2 = \frac{k}{m} - \frac{1}{4} \frac{c^2}{m^2}$

If  $c^2 = 4mk$  (critical damping):  $x = (a + bt)e^{-\mu t}$ ,  $\mu = \frac{c}{2m}$

If  $c^2 > 4mk$  (greater than critical damping):  $x = ae^{-\mu_1 t} + be^{-\mu_2 t}$

where  $\mu_1$  and  $\mu_2$  denote the following two values:

$$\mu_{1,2} = \frac{1}{2m} (c \pm \sqrt{c^2 - 4mk})$$

## 2. HARMONIC FORCED VIBRATIONS

With  $P = p \cos \omega_0 t$  in terms of arbitrary constants  $p$  and  $\omega_0$ :

$$x = a \cos \omega_0 t + b \sin \omega_0 t$$

$$\left[ (k - m\omega_0^2)^2 + c^2\omega_0^2 \right] a = (k - m\omega_0^2) p$$

$$\left[ (k - m\omega_0^2)^2 + c^2\omega_0^2 \right] b = c\omega_0 p$$

$$\left[ (k - m\omega_0^2)^2 + c^2\omega_0^2 \right] (a^2 + b^2) = p^2, \quad \frac{a}{b} = \frac{k - m\omega_0^2}{c\omega_0}$$

Thus  $a = 0$  and the vibration is in time quadrature relative to  $P$  when  $\omega_0 = \sqrt{k/m}$ , which is the value of  $\omega$  for undamped free vibration. The maximum amplitude or maximum of  $\sqrt{a^2 + b^2}$  for given  $p$ , however, occurs when

$$(d/d\omega_0) \left[ (k - m\omega_0^2)^2 + c^2\omega_0^2 \right] = 0$$

or when

$$\omega_o^2 = \frac{k}{m} - \frac{c^2}{2m^2}$$

This differs from  $k/m$  by twice as much as does  $\omega^2$  in a damped free oscillation.

These formulas exhibit several features for which analogs may reasonably be expected in more complicated cases, namely:

- (1) Two independent modes of damped free vibration occur. Their amplitudes can be chosen to make  $x$  and  $\dot{x}$  agree with any assumed initial values.
- (2) These free vibrations are oscillatory provided the damping constant  $c$  is not too large; in this case,  $c$  produces only a second-order change in the oscillatory frequency.
- (3) In a harmonic forced vibration,  $c$  introduces a component of  $x$  in time quadrature relative to the applied force  $P$  (proportional to  $\sin \omega_o t$  instead of to  $\cos \omega_o t$ ).
- (4)  $x$  is entirely in quadrature relative to  $P$  when the forcing frequency factor  $\omega_o$  equals the value of  $\omega$  for undamped free vibration.
- (5) The maximum amplitude of  $x$  for forcing at given  $p$ , when damping is present, occurs at an  $\omega_o$  differing from the undamped free  $\omega$  by more than does the oscillatory  $\omega$  in damped free vibration.

### 3. EXPERIMENTAL DETERMINATION OF $c$

If  $\mu \neq 0$ , its value can easily be determined from a curve showing either  $x$  or  $\ddot{x}$  as a function of  $t$  during damped free motion. Then  $c = 2m\mu$ .

If  $\omega$  is also determined from the curve, the ratio  $k/m$  can be calculated as  $k/m = \omega^2 + \mu^2$ . To determine  $k$  and  $m$  separately, one of them must be known from some other source.

Or, during a damped forced vibration the ratio  $b/a$  may be observed as the ratio of the components of  $x$  respectively in lagging quadrature to  $P$  and in phase with  $p$ , or the equal ratio for  $\ddot{x}$ . (Note that here  $\ddot{x} = -\omega_0^2 x$ ). Then

$$c = \frac{1}{\omega_0} (k - m\omega_0^2) \frac{b}{a}$$

In this case, the values of both  $k$  and  $m$  must be known.

### III. TWO-DIMENSIONAL VIBRATIONS

Assume that the kinetic energy  $T$  and potential energy  $V$  of a two-dimensional system can be written as <sup>4,5\*</sup>

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 + m_{12} \dot{x}\dot{y}, \quad V = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 + k_{12} xy$$

in which  $x$  and  $y$  are generalized coordinates and  $m_1$ ,  $m_2$ ,  $m_{12}$  are inertial and  $k_1$ ,  $k_2$ ,  $k_{12}$  elastic constants, of which only  $m_{12}$  and  $k_{12}$  may be negative. Substitution of first  $q = x$  and then  $q = y$  in Lagrange's equation or

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}} + \frac{\partial V}{\partial q} = Q_q$$

gives as equations of motion

$$m_1 \ddot{x}_1 + k_1 x + m_{12} \ddot{y} + k_{12} y = \bar{P}(t)$$

$$m_{12} \ddot{x}_1 + k_{12} x + m_2 \ddot{y} + k_2 y = \bar{Q}(t)$$

in which  $\bar{P}(t)$  and  $\bar{Q}(t)$  represent the total generalized forces acting on the

\* Also see Appendix A of Reference 1.

system (not including internal elastic forces). Part of  $\bar{P}$  and  $\bar{Q}$  may be due to linear damping forces. Expressing the latter in terms of damping constants  $c_1, c_2, c_{12}, c_{21}$ , the equations of motion may be written:

$$\begin{aligned} m_1 \ddot{x} + k_1 x + m_{12} \ddot{y} + k_{12} y + c_1 \dot{x} + c_{12} \dot{y} &= P(t) \\ m_{12} \ddot{x} + k_{12} x + m_2 \ddot{y} + k_2 y + c_{21} \dot{x} + c_2 \dot{y} &= Q(t) \end{aligned} \quad [2]$$

in which  $P$  and  $Q$  represent possible external forces acting on the system (aside from damping forces).

Certain restrictions on the possible values of the constants are worth noting. Let  $x$  and  $y$  be so chosen that  $T$  and  $V$  are never negative. Damping effects can never increase the sum  $T + V$ . Multiply the first of Equations [2] by  $\dot{x}$  and the second by  $\dot{y}$  and add the two equations. The sum of the resulting  $m$  and  $k$  terms is easily seen to equal  $(d/dt) (T + V)$ ; hence, if

$$P = Q = 0$$

$$\frac{d}{dt} (T + V) = -c_1 \dot{x}^2 - c_2 \dot{y}^2 - (c_{12} + c_{21}) \dot{x} \dot{y}$$

To keep  $(d/dt) (T + V)$  from ever being positive, it is necessary that  $c_1 \geq 0$ ,  $c_2 \geq 0$ , since either  $\dot{x}$  or  $\dot{y}$  may vanish. Similarly, to keep  $T \geq 0$  and  $V \geq 0$ , it is necessary that  $m_1, m_2, k_1$ , and  $k_2$  all be  $\geq 0$ .

Further restrictions may be inferred from the following theorem. Let  $\alpha, \beta, \gamma, e, g$  be real numbers. Then

$$\alpha e^2 + \beta g^2 + \gamma e g \geq 0 \text{ or } \alpha e^2 + \beta g^2 \geq -\gamma e g \quad [3]$$

for all values of  $e$  and  $g$  if and only if

$$\alpha \geq 0, \beta \geq 0, \gamma^2 \leq 4\alpha\beta$$

To prove this, note first that  $\alpha$  and  $\beta$  cannot be negative because of

cases in which only  $e = 0$  or  $g = 0$ . Relation [3] then clearly holds if  $e$  and  $g$  are such that  $\gamma eg \geq 0$ .

Suppose, however, that  $\gamma eg < 0$ . Then Equation [3] in its second form is equivalent to the following:

$$(\alpha e^2 + \beta g^2)^2 \geq (\gamma eg)^2 \quad [3a]$$

provided that positive square roots are taken in passing back from Equation [3a] to Equation [3]. But

$$(\alpha e^2 + \beta g^2)^2 = (\alpha e^2 - \beta g^2)^2 + 4\alpha\beta(eg)^2$$

Hence, if  $\alpha > 0$  and  $\beta > 0$  and if  $e$  and  $g$  are chosen so that  $\alpha e^2 = \beta g^2$ , then  $(\alpha e^2 + \beta g^2)^2 = 4\alpha\beta(eg)^2$ . Thus Equation [3a] can hold generally only if  $4\alpha\beta \geq \gamma^2$ . If either  $\alpha$  or  $\beta$  vanishes, Equation [3] requires that  $\gamma = 0$ . Conversely, if the condition that  $4\alpha\beta \geq \gamma^2$  is met but  $\alpha e^2 \neq \beta g^2$ , then  $(\alpha e^2 + \beta g^2)^2 > 4\alpha\beta(eg)^2 > \gamma^2(eg)^2$  and Equation [3a] holds, also Equation [3].

Substitute here  $\alpha = m_1/2$ ,  $\beta = m_2/2$ ,  $\gamma = m_{12}$ ,  $e = \dot{x}$  and  $g = \dot{y}$ ; next,  $\alpha = k_1/2$ ,  $\beta = k_2/2$ ,  $\gamma = k_{12}$ ,  $e = x$  and  $g = y$ ; and finally  $\alpha = c_1$ ,  $\beta = c_2$ ,  $\gamma = c_{12} + c_{21}$ ,  $e = \dot{x}$  and  $g = \dot{y}$ . Compare the resulting expressions with expressions previously written for  $T$ ,  $V$ , and  $(d/dt)(T + V)$ . It will then be clear that, to prevent  $T$  and  $V$  from ever becoming negative or  $(d/dt)(T + V)$  positive, it is necessary and sufficient that

$$m_{12}^2 \leq m_1 m_2, \quad k_{12}^2 \leq k_1 k_2, \quad (c_{12} + c_{21})^2 \leq 4c_1 c_2 \quad [4]$$

These restrictions will be assumed to hold.

It follows then also that

$$2m_{12}k_{12} \leq m_1 k_2 + m_2 k_1, \quad c_{12}c_{21} \leq c_1 c_2 \quad [5a, b]$$



For  $(m_1 k_2 + m_2 k_1)^2 = (m_1 k_2 - m_2 k_1)^2 + 4m_1 m_2 k_1 k_2 \geq 4m_1 m_2 k_1 k_2 \geq 4m_{12}^2 k_{12}^2$  by relations [4]. (Note that a square cannot be negative.) Similarly, in any case  $4c_{12} c_{21} \leq 4c_{12} c_{21} + (c_{12} - c_{21})^2 = (c_{12} + c_{21})^2$ ; hence, by Equation [4],  $4c_{12} c_{21} \leq 4c_1 c_2$ .

Two other relations that can be inferred in a similar way from relations [4] are:

$$(c_{12} + c_{21})m_{12} \leq c_1 m_2 + c_2 m_1, \quad (c_{12} + c_{21})k_{12} \leq c_1 k_2 + c_2 k_1 \quad [5c, d]$$

# 1. UNDAMPED FREE VIBRATIONS

Undamped free oscillations merits consideration as background for study of the damped case. Let  $c_1 = c_2 = c_{12} = c_{21} = 0$ ,  $P = Q = 0$ . Then Equations [2] become

$$m_1 \ddot{x} + k_1 x + m_{12} \ddot{y} + k_{12} y = 0 \quad m_{12} \ddot{x} + k_{12} x + m_2 \ddot{y} + k_2 y = 0 \quad [6]$$

Two special cases may first be noted. According to Equations [6],  $x$  can vibrate while  $y = 0$  only if  $k_1 - m_1 \omega^2$  and  $k_{12} - m_{12} \omega^2$  are both zero. The first condition fixes  $\omega$  at  $\sqrt{k_1/m_1}$ ; the second requires that either  $m_{12} = k_{12} = 0$  or  $m_1 k_{12} = m_{12} k_1$ . Similarly,  $y$  can vibrate with  $x = 0$ , and  $\omega = \sqrt{k_2/m_2}$  only if either  $m_{12} = k_{12} = 0$  or  $m_2 k_{12} = m_{12} k_2$ .

If  $x$  and  $y$  vibrate together in proportion to  $\cos \omega t$ , the following equations must be satisfied:

$$\begin{aligned} (k_1 - m_1 \omega^2) x + (k_{12} - m_{12} \omega^2) y &= 0 \\ (k_{12} - m_{12} \omega^2) x + (k_2 - m_2 \omega^2) y &= 0 \end{aligned}$$

Elimination of  $x$  and  $y$  gives for the determination of  $\omega$  the following equation:

$$(k_1 - m_1 \omega^2)(k_2 - m_2 \omega^2) - (k_{12} - m_{12} \omega^2)^2 = 0 \quad [7a]$$

or

$$(m_1 m_2 - m_{12}^2) \omega^4 - (m_1 k_2 + m_2 k_1 - 2m_{12} k_{12}) \omega^2 + (k_1 k_2 - k_{12}^2) = 0 \quad [7b]$$

If  $k_{12}^2 = k_1 k_2$ , one root of Equations [7b] is:  $\omega^2 = 0$ . Alternatively, if  $m_1 m_2 = m_{12}^2$ , only one mode of vibration is possible.

Assume now that  $k_{12}^2 < k_1 k_2$  and  $m_{12}^2 < m_1 m_2$ . To locate  $\omega^2$ , consider L, the left-hand member of Equation [7a] or Equation [7b], as a function of  $\omega^2$ . At  $\omega^2 = 0$ ,  $L > 0$ ; but when  $\omega^2$  has increased to  $\omega_{\min}^2$  representing the lesser of the two values  $k_1/m_1$  and  $k_2/m_2$ , then it is clear from Equation [7a] that  $L < 0$ . Hence  $L = 0$  at some positive value of  $\omega^2$  less than  $\omega_{\min}^2$ . Also at the greater of the values  $k_1/m_1$  and  $k_2/m_2$ ,  $L < 0$ , but as  $\omega^2 \rightarrow \infty$  it is clear from Equations [7b] that  $L > 0$ . Hence a second root of Equation [7a,b] occurs at a value of  $\omega^2$  greater than both  $k_1/m_1$  and  $k_2/m_2$ .

Thus two different modes of vibration of the system are possible with both x and y vibrating. In each mode

$$\frac{y}{x} = -\frac{k_{12} - m_{12} \omega^2}{k_2 - m_2 \omega^2} = -\frac{k_1 - m_1 \omega^2}{k_{12} - m_{12} \omega^2}$$

## 2. DAMPED FREE VIBRATIONS

Let  $P = Q = 0$  so that Equations [2] read

$$m_1 \ddot{x} + k_1 x + m_{12} \ddot{y} + k_{12} y + c_1 \dot{x} + c_{12} \dot{y} = 0 \quad [8a]$$

$$m_{12} \ddot{x} + k_{12} x + m_2 \ddot{y} + k_2 y + c_{21} \dot{x} + c_2 \dot{y} = 0 \quad [8b]$$

In special cases especially if  $m_{12} = k_{12} = 0$  and  $c_{12} = 0$  so that Equation [8a] reduces to Equation [1] with  $P = 0$ , x can vary while  $y = 0$ ; or,

similarly, if  $c_{21} = 0$ ,  $y$  alone may vary. Such cases will not be discussed further here.

For the general case, solutions may be sought in which<sup>6</sup>

$$x = ae^{\lambda t}, \quad y = be^{\lambda t}$$

where  $a$ ,  $b$ , and  $\lambda$  are non-zero constants, real or complex. Substituting in Equations [8a,b] and canceling out  $e^{\lambda t}$ :

$$(m_1 \lambda^2 + k_1 + c_1 \lambda) a + (m_{12} \lambda^2 + k_{12} + c_{12} \lambda) b = 0 \quad [9a]$$

$$(m_{12} \lambda^2 + k_{12} + c_{21} \lambda) a + (m_2 \lambda^2 + k_2 + c_2 \lambda) b = 0 \quad [9b]$$

The result of eliminating  $a$  and  $b$  from these equations may be written:

$$\epsilon_4 \lambda^4 + \epsilon_3 \lambda^3 + \epsilon_2 \lambda^2 + \epsilon_1 \lambda + \epsilon_0 = 0 \quad [10]$$

where

$$\epsilon_0 = k_1 k_2 - k_{12}^2, \quad \epsilon_1 = c_1 k_2 + c_2 k_1 - (c_{12} + c_{21}) k_{12}$$

$$\epsilon_2 = m_1 k_2 + m_2 k_1 - 2m_{12} k_{12} + c_1 c_2 - c_{12} c_{21}$$

$$\epsilon_3 = c_1 m_2 + c_2 m_1 - (c_{12} + c_{21}) m_{12}, \quad \epsilon_4 = m_1 m_2 - m_{12}^2$$

The coefficients  $\epsilon_0 \dots \epsilon_4$  are all  $\geq 0$ , according to Equations [4] and [5a,b,c,d]. Hence no root  $\lambda$  of Equation [10] can be a positive real number. Probably if the damping is strong enough, negative real roots may occur, possibly even four in number, but this difficult question is of little practical interest here.

For the general case, write  $\lambda = -\mu + i\omega$  where  $i = \sqrt{-1}$  and  $\mu$  and  $\omega$  are real numbers. The following two equations result from substituting in Equation [10], then equating the real and imaginary parts separately to zero,

and dividing the imaginary equation by  $i\omega$  on the assumption that  $\omega \neq 0$ :

$$\epsilon_4 \omega^4 - (\epsilon_2 - 3\epsilon_3\mu + 6\epsilon_4\mu^2)\omega^2 + \epsilon_0 - \epsilon_1\mu + \epsilon_2\mu^2 - \epsilon_3\mu^3 + \epsilon_4\mu^4 = 0 \quad [11a]$$

$$\epsilon_1 - \epsilon_3\omega^2 - (2\epsilon_2 - 4\epsilon_4\omega^2)\mu + 3\epsilon_3\mu^2 - 4\epsilon_4\mu^3 = 0 \quad [11b]$$

These equations determine  $\mu$  and  $\omega^2$ . The conjugate quantity  $-\mu - i\omega$  is then also a root of Equation [10]. Since there are only four roots in all, there can be only two pairs of values,  $\mu_1$  and  $\omega_1$ , and  $\mu_2$  and  $\omega_2$ . These pairs define two modes of damped oscillation. Since damping cannot increase the total energy, it must turn out that both  $\mu_1$  and  $\mu_2$  are positive.

To obtain real expressions either the real parts of all quantities (i.e., solutions) may be chosen or the imaginary parts divided by  $i$ ; the two pairs of real solutions thus obtained are in relative time quadrature. The value of the ratio  $b/a$  for each mode may be obtained from Equations [9a, b]. Since usually  $b/a$  will turn out complex, there will generally be a difference of phase between  $x$  and  $y$  as functions of the time.

Thus four real expressions are obtained representing four independent damped oscillations. For these oscillations,  $x$  and  $y$  can be written thus:

$$x = e^{-\mu_1 t} (A_1 \cos \omega_1 t + A'_1 \sin \omega_1 t), \quad y = r_1 e^{-\mu_1 t} [A_1 \cos (\omega_1 t + \epsilon_1) + A'_1 \sin (\omega_1 t + \epsilon_1)]$$

or

$$y = e^{-\mu_2 t} (A_2 \cos \omega_2 t + A'_2 \sin \omega_2 t), \quad y = r_2 e^{-\mu_2 t} [A_2 \cos (\omega_2 t + \epsilon_2) + A'_2 \sin (\omega_2 t + \epsilon_2)]$$

Here  $A_1, A'_1, A_2, A'_2$  are independent arbitrary constants which can be adjusted to fit any assumed initial values of  $x, \dot{x}, y, \dot{y}$ . It should be noted that

$$\ddot{x} = -(\omega_1^2 - \mu_1^2)x \text{ and } \ddot{y} = -(\omega_1^2 - \mu_1^2)y$$

in any one mode whereas in the other

$$\ddot{x} = -(\omega_2^2 - \mu_2^2)x \text{ and } \ddot{y} = -(\omega_2^2 - \mu_2^2)y$$

Only small damping effects appear to be important in practice. Hence no general discussion of Equations [11a,b] will be undertaken here.

If the c's are sufficiently small,  $\mu$  will also be small, and the coefficients  $\epsilon_1$  and  $\epsilon_3$  are likewise small. Consequently all terms in Equation [11a] containing  $\mu$  are small at least to the second order, and the last three terms in Equation [11b] are small to the third order. For an approximate solution, these terms may all be dropped. Then Equation [11a] becomes:  $\epsilon_4 \omega^4 - \epsilon_2 \omega^2 + \epsilon_0 = 0$ . This agrees with Equations [7a,b] for the case of no damping so that to the degree of approximation under discussion, the oscillation frequencies are the same as if there were no damping. From Equation [11b] the approximate value of  $\mu$  is

$$\mu = \frac{1}{2} \frac{\epsilon_1 - \epsilon_3 \omega^2}{\epsilon_2 - 2 \epsilon_4 \omega^2} \quad [12]$$

More accurate solutions can be obtained from Equations [11a,b] by a process of successive approximation.

### 3. HARMONIC FORCED VIBRATIONS

If the applied forces are harmonic functions of the time  $t$ , they cause harmonic vibrations of  $x$  and  $y$ . At the start there may also exist superposed damped free oscillations whose amplitudes can be adjusted so as to produce

on the whole any initial values of  $x$ ,  $y$ ,  $\dot{x}$ ,  $\dot{y}$ . These damped free oscillations will be assumed to have died out.

Since in the one-dimensional case, the presence of damping introduces a phase difference, assume:

$$\begin{aligned} P &= p \cos \omega_0 t + p' \sin \omega_0 t, & Q &= q \cos \omega_0 t + q' \sin \omega_0 t \\ x &= a_1 \cos \omega_0 t + a'_1 \sin \omega_0 t, & y &= a_2 \cos \omega_0 t + a'_2 \sin \omega_0 t \end{aligned} \quad [13]$$

In Equations [2] the  $\cos \omega_0 t$  and  $\sin \omega_0 t$  terms must balance separately.

After canceling the time factors, the result is the following four equations:

$$\begin{aligned} (k_1 - m_1 \omega_0^2) a_1 + c_1 \omega_0 a'_1 + (k_{12} - m_{12} \omega_0^2) a_2 + c_{12} \omega_0 a'_2 &= p \\ -c_1 \omega_0 a_1 + (k_1 - m_1 \omega_0^2) a'_1 - c_{12} \omega_0 a_2 + (k_{12} - m_{12} \omega_0^2) a'_2 &= p' \\ (k_{12} - m_{12} \omega_0^2) a_1 + c_{21} \omega_0 a'_1 + (k_2 - m_2 \omega_0^2) a_2 + c_2 \omega_0 a'_2 &= q \\ -c_{21} \omega_0 a_1 + (k_{12} - m_{12} \omega_0^2) a'_1 - c_2 \omega_0 a_2 + (k_2 - m_2 \omega_0^2) a'_2 &= q' \end{aligned} \quad [14]$$

Here  $p$ ,  $p'$ ,  $q$ ,  $q'$ ,  $a_1$ ,  $a'_1$ ,  $a_2$ ,  $a'_2$  are eight real numbers. In general, any four of them can be assigned arbitrarily; the equations then fix the values of the other four. Furthermore, since  $\cos \omega_0 t$  and  $\sin \omega_0 t$  differ only in phase, the zero for  $t$  can be so adjusted that any chosen one of the eight quantities  $a_1 \dots q'$  vanishes, without altering the physical form of the vibration. Thus all cases can be covered while keeping one coefficient zero.

In particular, Equations [14] may be solved for the amplitudes  $a_1$ ,  $a'_1$ ,  $a_2$ ,  $a'_2$  caused by given applied forces represented by  $p$ ,  $p'$ ,  $q$ ,  $q'$ . The determinant  $\Delta$  of the coefficients of  $a_1$ ,  $a'_1$ ,  $a_2$ ,  $a'_2$  is easily found to have

the value

$$\Delta = \left[ (k_1 - m_1 \omega_o^2)(k_2 - m_2 \omega_o^2) - (k_{12} - m_{12} \omega_o^2)^2 \right] - \omega_o^4 (c_1 c_2 - c_{12} c_{21})^2$$

If there is no damping, comparison with Equation [7a] shows that  $\Delta = 0$

when  $\omega_o$  equals the value of  $\omega$  for either of the frequencies of undamped free vibration of the system.

If  $c_1, c_2, c_{12}, c_{21}$  are merely all small,  $\Delta$  will vanish at two slightly modified frequencies that differ also slightly from the frequencies of damped free oscillation. As  $\omega_o$  approaches either of these frequencies at which  $\Delta = 0$  while  $p, p', q, q'$  remain fixed, the amplitude of the forced vibration becomes large (the phenomenon called resonance).

#### 4. EXPERIMENTAL DETERMINATION OF $c_1, c_2, c_{12}, c_{21}$

One method is to make "bumping" observations by starting a motion and recording it as it decays. By proper adjustment of the initial values of  $x, \dot{x}, y, \dot{y}$ , the system can be made to vibrate in either of its two modes of damped free vibration with the other mode absent. Observations may be made of either  $x$  and  $y$  or  $\ddot{x}$  and  $\ddot{y}$  as functions of the time since  $\ddot{x} = -(\omega_1^2 - \mu_1^2)x$  and  $\ddot{y} = -(\omega_1^2 - \mu_1^2)y$  in one mode and  $\ddot{x} = -(\omega_2^2 - \mu_2^2)x, \ddot{y} = -(\omega_2^2 - \mu_2^2)y$  in the other. From these observations, values can be calculated for each mode of the frequency  $\omega$ , the damping constant  $\mu$ , and the amplitude ratio  $r$  and phase  $\epsilon$  of  $y$  relative to  $x$ , giving the eight known quantities

$$\omega_1 \omega_2 \mu_1 \mu_2 r_1 r_2 \epsilon_1 \epsilon_2$$

Insertion of  $\omega_1$  and  $\mu_1$  and then of  $\omega_2$  and  $\mu_2$  for  $\omega$  and  $\mu$  in Equations [11a,b] then provides four equations which can be solved numerically for  $c_1, c_2, c_{12}$ , and  $c_{21}$  provided the six constants  $m_1, m_2, m_{12}, k_1, k_2$ , and  $k_{12}$  are known. It might be more accurate, however, to use equations containing the constants  $\epsilon_1$  and  $\epsilon_2$  which differ from zero only because of damping. If bumping observations are to be used, further study of the methods of calculation should be made. The damping may be weak enough to justify the use of simplifying approximations.

It may be worth noting that observation of all eight quantities  $\omega_1$  to  $\epsilon_2$  should make possible the calculation of nine of the ten quantities  $m_1, m_2, m_{12}, k_1, k_2, k_{12}, c_1, c_2, c_{12}$ , and  $c_{21}$ . For a restriction exists on the possible variation of these quantities. Let Equations [8a,b] be multiplied by an arbitrary constant  $\underline{s}$ . The new equations may then be regarded either as equations in a different form for the original system or as equations for a different system having constants  $\underline{s}$  times as great but the same damping modes as the original system. In order to know which system of this similitude class the observed constants  $\omega_1 . . . . . \epsilon_2$  refer to, it is necessary to know at least one of the ten quantities  $m_1 . . . . . c_{21}$ . Then the remaining nine can all be calculated from the eight observed constants  $\omega_1 . . . . . \epsilon_2$ .

(If Equations [8a,b] are multiplied by different numbers, they are still valid for the original system but cannot be regarded as equations in the same form as Equations [8a,b] for a different system because the new  $m_{12} \neq m_{21}$



and  $k_{12} \neq k_{21}$ .) Even if the initial values of  $x, \dot{x}, y, \dot{y}$  cannot be properly adjusted, since one mode will usually die out before the other, both sets of values,  $\mu_1$  and  $\omega_1$  and  $\mu_2$  and  $\omega_2$ , can be inferred from the same curve of  $x$  or  $y$  as a function of time. If both modes persist, it is still possible to observe each mode in turn by means of a filter.<sup>7-10</sup> Or a vibrator may be used and adjusted in frequency so as to be in resonance with one mode; then, after the vibrator is removed, a damped free oscillation will occur in this mode only.

If  $c_{12} = c_{21} = 0$ ,  $c_1$  and  $c_2$  can be calculated from  $\mu_1$  and  $\mu_2$ . Otherwise the observed values of  $\mu_1$  and  $\mu_2$  furnish only two relations among the four quantities  $c_1, c_2, c_{12}, c_{21}$ .

An alternative method is to study forced harmonic vibrations produced by applied forces  $P$  and  $Q$  whose relative amplitudes and phases can be controlled. (Applied forces are pure  $P$  when they do no work during variation of  $y$  alone, or pure  $Q$  when no work is done during variation of  $x$  alone.) Two alternative procedures will be described which require no measurements of  $P$  or  $Q$ . The constants  $m_1, m_2, m_{12}, k_1, k_2, k_{12}$ , however, must be known. Either  $x$  and  $y$  or  $\ddot{x}$  and  $\ddot{y}$  may be observed since in forced oscillations  $\ddot{x} = -\omega_0^2 x$ ,  $\ddot{y} = -\omega_0^2 y$  and  $\omega_0^2$  will be seen to cancel out in all final formulas.

First Procedure: Isolation of  $c_1, c_2, c_{12}, c_{21}$  in turn. Make observations as follows:

- (1) Cause  $x$  to vibrate with  $y = 0$ . Assume  $p' = 0$ , so that  $a_1$  denotes

the amplitude of the component of  $x$  that is in phase with  $P$  and  $a'_1$  the amplitude of the quadrature component of  $x$ . To do this, apply  $P = p \cos \omega_0 t$  and adjust the amplitude and phase of  $Q$  so that  $a_2 = a'_2 = 0$ . Then Equations [14] reduce to

$$\begin{aligned} (k_1 - m_1 \omega_0^2) a_1 + c_1 \omega_0 a'_1 &= p \\ (k_{12} - m_{12} \omega_0^2) a_1 + c_{21} \omega_0 a'_1 &= q \end{aligned} \quad \left\| \begin{aligned} -c_1 \omega_0 a_1 + (k_1 - m_1 \omega_0^2) a'_1 &= 0 \\ -c_{21} \omega_0 a_1 + (k_{12} - m_{12} \omega_0^2) a'_1 &= q' \end{aligned} \right.$$

The magnitude of  $\omega_0$  should be made quite different from  $\sqrt{k_1/m_1}$ . Only the ratio  $a'_1/a_1$  needs to be observed.

Probably the adjustment of  $Q$  can be effected most conveniently by varying its amplitude  $\sqrt{(q)^2 + (q')^2}$  until  $a_2$  (or the component of  $y$  in phase with  $P$ ) is zero, then varying the phase of  $Q$  (thus varying  $q'$ ) until the quadrature amplitude  $a'_2$  of  $y$  equals zero, and repeating these adjustments in turn until both  $a_2$  and  $a'_2$  remain negligibly small.

Then

$$c_1 = \frac{a'_1}{\omega_0 a_1} (k_1 - m_1 \omega_0^2)$$

(2) Similarly, to keep  $x = 0$ , apply  $Q = q \cos \omega_0 t$ , hence  $q' = 0$ , and with  $\omega_0$  not near  $\sqrt{k_2/m_2}$ , adjust  $p$  and  $p'$  so that  $a_1 = a'_1 = 0$ , and read  $a'_2/a_2$ . Then

$$c_2 = \frac{a'_2}{\omega_0 a_2} (k_2 - m_2 \omega_0^2)$$

(3) Cause  $x$  and  $y$  to vibrate in phase with  $P$ ; that is, writing  $P = p \cos \omega_0 t$  with  $p' = 0$ , adjust  $q$  and  $q'$  so that  $a'_1 = a'_2 = 0$ . Read  $a_1/a_2$ . Then from the second one of Equations [14]

$$c_{12} = -\frac{a_1}{a_2} c_1$$

In this case the simplest way to effect the required adjustment of  $Q$  might be to vary its amplitude so as to reduce the larger of  $a'_1$  and  $a'_2$  until  $a'_1 = a'_2$ , then adjust the phase of  $Q$  so as to minimize  $a'_2$ , and repeat these adjustments until  $a'_1$  and  $a'_2$  have been made sufficiently small.

(4) Cause  $x$  and  $y$  to vibrate in phase with  $Q$ , assuming  $q' = 0$ . Adjust  $p$  and  $p'$  so that  $a'_1 = a'_2 = 0$  nearly enough. Read  $a_2/a_1$ . Then from the fourth of Equations [14]

$$c_{21} = -\frac{a_2}{a_1} c_2$$

This procedure should yield the most accurate values of the four  $c$ 's, but the experimental adjustments required may be considered too tedious.

Second Procedure: Single-phase forcing. Apply  $P$  and  $Q$  in any known ratio but in the same phase. Write  $P = p \cos \omega_0 t$ ,  $Q = q \cos \omega_0 t$ , so that  $p' = q' = 0$ . Read  $a_1, a_2$  as amplitudes of inphase and  $a'_1, a'_2$  as amplitudes of quadrature components of  $x$  and  $y$ . Repeat with a different ratio  $Q/P$ , distinguishing the amplitudes thus obtained by a bar.

Substitute each set of  $a$ 's in turn into the second and fourth of Equations [14], in which  $p' = q' = 0$ . The resulting equations can be

written:

$$\omega_o a_1 c_1 + \omega_o a_2 c_{12} = (k_1 - m_1 \omega_o^2) a_1' + (k_{12} - m_{12} \omega_o^2) a_2'$$

$$\omega_o \bar{a}_1 c_1 + \omega_o \bar{a}_2 c_{12} = (k_1 - m_1 \omega_o^2) \bar{a}_1' + (k_{12} - m_{12} \omega_o^2) \bar{a}_2'$$

$$\omega_o a_1 c_{21} + \omega_o a_2 c_2 = (k_{12} - m_{12} \omega_o^2) a_1' + (k_2 - m_2 \omega_o^2) a_2'$$

$$\omega_o \bar{a}_1 c_{21} + \omega_o \bar{a}_2 c_2 = (k_{12} - m_{12} \omega_o^2) \bar{a}_1' + (k_2 - m_2 \omega_o^2) \bar{a}_2'$$

These two pairs of equations are easily solved for  $c_1$ ,  $c_{12}$ , and  $c_2$ ,  $c_{21}$ .

#### IV. THREE-DIMENSIONAL VIBRATIONS

Let  $x$ ,  $y$ ,  $z$  denote the three displacement variables, for example  $v$ ;  $\gamma$   $\propto$  motion of a rudder (see Reference 1). Then linear equations of motion can be written as follows:

$$m_1 \ddot{x} + k_1 x + m_{12} \ddot{y} + k_{12} y + m_{13} \ddot{z} + k_{13} z + c_1 \dot{x} + c_{12} \dot{y} + c_{13} \dot{z} = P(t) \quad [15a]$$

$$m_{12} \ddot{x} + k_{12} x + m_2 \ddot{y} + k_2 y + m_{23} \ddot{z} + k_{23} z + c_{21} \dot{x} + c_2 \dot{y} + c_{23} \dot{z} = Q(t) \quad [15b]$$

$$m_{13} \ddot{x} + k_{13} x + m_{23} \ddot{y} + k_{23} y + m_3 \ddot{z} + k_3 z + c_{31} \dot{x} + c_{32} \dot{y} + c_3 \dot{z} = R(t) \quad [15c]$$

Here  $P$ ,  $Q$ , and  $R$  are generalized external forces so defined that the rate at which they do work on the system is always  $P\dot{x} + Q\dot{y} + R\dot{z}$ . The  $m$ 's are of the nature of inertial constants and the  $k$ 's of elastic constants.

Then there may be, as in Equations [15a,b,c], nine linear damping constants  $c_1$ ,  $c_2$ ,  $c_3$ ,  $c_{12}$ ,  $c_{13}$ ,  $c_{21}$ ,  $c_{23}$ ,  $c_{31}$ ,  $c_{32}$ . The six cross constants  $c_{12}$ , etc., will be limited in relative size, as in the two-dimensional case, since the damping necessarily tends to decrease the total

energy  $T + V$ ; they are likely to be relatively small and may be negligible, but this cannot be assumed to be true in general because the magnitudes of all nine constants will vary with the choice of the variables to be called  $x$ ,  $y$ ,  $z$ .

The situation will be analogous in general to that for two dimensions. If  $P = Q = R = 0$  and all  $c$ 's are zero, there will be solutions of Equations [15a,b,c] representing three modes of undamped free vibration. If any  $c$ 's do not vanish, these modes become three modes of damped free oscillation; or, if the  $c$ 's are sufficiently large, one or more modes may be replaced by two modes of exponential decrease without oscillation, such as were represented by formulas in the one-dimensional case.

In the oscillatory case, on the other hand, there will be three damping constants  $\mu_1, \mu_2, \mu_3$ . In any one mode of damped oscillation, the three variables  $x$ ,  $y$ , and  $z$  may be assumed to be proportional to  $e^{-\mu_1 t} \cos(\omega_1 t + \epsilon)$ , in another mode to  $e^{-\mu_2 t} \cos(\omega_2 t + \epsilon)$ , and in the third to  $e^{-\mu_3 t} \cos(\omega_3 t + \epsilon)$ , the phase angle  $\epsilon$  being different in general for  $x$ ,  $y$  and  $z$  and different in the three modes.

The frequency factors  $\omega_1, \omega_2, \omega_3$  will not be quite the same as in the undamped vibrations, but the difference will be only of the second order if the damping is relatively small.

A more detailed discussion of these various cases follows:

## 1. UNDAMPED FREE VIBRATIONS

If  $P = Q = R = 0$  and all the  $c$ 's are zero, a solution of Equations [15a,b,c] is

$x = a_1 \cos \omega t$ ,  $y = a_2 \cos \omega t$ , and  $z = a_3 \cos \omega t$ ,  $a_1$ ,  $a_2$ , and  $a_3$  being real numbers; from Equations [15a,b,c] :

$$(k_1 - m_1 \omega^2) a_1 + (k_{12} - m_{12} \omega^2) a_2 + (k_{13} - m_{13} \omega^2) a_3 = 0$$

$$(k_{12} - m_{12} \omega^2) a_1 + (k_2 - m_2 \omega^2) a_2 + (k_{23} - m_{23} \omega^2) a_3 = 0$$

$$(k_{13} - m_{13} \omega^2) a_1 + (k_{23} - m_{23} \omega^2) a_2 + (k_3 - m_3 \omega^2) a_3 = 0$$

Equating the determinant of  $a_1$ ,  $a_2$ ,  $a_3$  in these equations to zero gives the equation:

$$\begin{aligned} & (k_1 - m_1 \omega^2)(k_2 - m_2 \omega^2)(k_3 - m_3 \omega^2) \\ & + 2(k_{12} - m_{12} \omega^2)(k_{13} - m_{13} \omega^2)(k_{23} - m_{23} \omega^2) \\ & - (k_1 - m_1 \omega^2)(k_{23} - m_{23} \omega^2)^2 - (k_2 - m_2 \omega^2)(k_{13} - m_{13} \omega^2)^2 \\ & - (k_3 - m_3 \omega^2)(k_{12} - m_{12} \omega^2)^2 = 0 \end{aligned} \quad [16]$$

This is a cubic equation in  $\omega^2$  whose three roots furnish the frequencies for three modes of undamped free vibration. Any two of the original equations can be solved for the ratios of  $a_1$ ,  $a_2$ , and  $a_3$  to each other in any one of the three modes (see, for example, Appendix C of Reference 1).

## 2. DAMPED FREE VIBRATIONS

Assume  $P = Q = R = 0$  and write

$$x = a_1 e^{\lambda t}, \quad y = a_2 e^{\lambda t}, \quad z = a_3 e^{\lambda t}$$

where  $a_1, a_2, a_3$  and  $\lambda$  may all be complex numbers. Substitution in Equations [15a,b,c] then gives:

$$(k_1 + m_1 \lambda^2 + c_1 \lambda) a_1 + (k_{12} + m_{12} \lambda^2 + c_{12} \lambda) a_2 + (k_{13} + m_{13} \lambda^2 + c_{13} \lambda) a_3 = 0$$

$$(k_{12} + m_{12} \lambda^2 + c_{21} \lambda) a_1 + (k_2 + m_2 \lambda^2 + c_2 \lambda) a_2 + (k_{23} + m_{23} \lambda^2 + c_{23} \lambda) a_3 = 0$$

$$(k_{13} + m_{13} \lambda^2 + c_{31} \lambda) a_1 + (k_{23} + m_{23} \lambda^2 + c_{32} \lambda) a_2 + (k_3 + m_3 \lambda^2 + c_3 \lambda) a_3 = 0$$

The determinant of  $a_1, a_2, a_3$  in these three equations set equal to zero gives :

$$\begin{aligned} & (k_1 + m_1 \lambda^2 + c_1 \lambda)(k_2 + m_2 \lambda^2 + c_2 \lambda)(k_3 + m_3 \lambda^2 + c_3 \lambda) \\ & + (k_{12} + m_{12} \lambda^2 + c_{12} \lambda)(k_{23} + m_{23} \lambda^2 + c_{23} \lambda)(k_{13} + m_{13} \lambda^2 + c_{31} \lambda) \\ & + (k_{12} + m_{12} \lambda^2 + c_{21} \lambda)(k_{23} + m_{23} \lambda^2 + c_{32} \lambda)(k_{13} + m_{13} \lambda^2 + c_{13} \lambda) \\ & - (k_1 + m_1 \lambda^2 + c_1 \lambda)(k_{23} + m_{23} \lambda^2 + c_{23} \lambda)(k_{23} + m_{23} \lambda^2 + c_{32} \lambda) \\ & - (k_2 + m_2 \lambda^2 + c_2 \lambda)(k_{13} + m_{13} \lambda^2 + c_{13} \lambda)(k_{13} + m_{13} \lambda^2 + c_{31} \lambda) \\ & - (k_3 + m_3 \lambda^2 + c_3 \lambda)(k_{12} + m_{12} \lambda^2 + c_{12} \lambda)(k_{12} + m_{12} \lambda^2 + c_{21} \lambda) = 0 \end{aligned}$$

[17]

This is an equation of the sixth degree in  $\lambda$ . It may have real roots if the  $c$ 's are large enough, perhaps as many as six real roots. On the other hand, analogy with the two-dimensional case suggests that if the  $c$ 's are not too large, there will be six complex roots in three pairs:  $-\mu_1 \pm i\omega_1$ ,  $-\mu_2 \pm i\omega_2$ ,  $-\mu_3 \pm i\omega_3$ .

Two equations for the determination of  $\omega_1, \omega_2, \omega_3$  and  $\mu_1, \mu_2, \mu_3$ , analogous to Equations [11a, b] in two dimensions, can be obtained by substituting  $\lambda = -\mu + i\omega$  and separating real and imaginary parts. In the three-dimensional case, however, these equations are voluminous and the chance of their ever being put to practical use seems to be very small, hence they will not be written out here in full.

For practical use when the  $c$ 's and hence also the  $\mu$ 's are small, abbreviated approximate equations can be obtained by omitting all terms of second or higher order, that is, all terms containing a power of  $\mu$  higher than the first or both  $\mu$  and one of the  $c$ 's or the product of two  $c$ 's. This rule of approximation justifies replacing  $\lambda^2$  in Equation [17] by  $-\omega^2 - 2i\omega\mu$  and also  $\lambda$  by  $i\omega$ . Furthermore, all products of  $c$  terms may be omitted. The first of the six products in Equation [17], for example, is to be replaced by

$$(k_1 - m_1\omega^2 - 2i\omega m_1\mu + i\omega c_1)(k_2 - m_2\omega^2 - 2i\omega m_2\mu + i\omega c_2)$$

$$(k_3 - m_3\omega^2 - 2i\omega m_3\mu + i\omega c_3)$$

and then expanded as



$$\begin{aligned}
& (k_1 - m_1 \omega^2)(k_2 - m_2 \omega^2)(k_3 - m_3 \omega^2) \\
& + i\omega (-2m_1 \mu + c_1)(k_2 - m_2 \omega^2)(k_3 - m_3 \omega^2) \\
& + i\omega (-2m_2 \mu + c_2)(k_1 - m_1 \omega^2)(k_3 - m_3 \omega^2) \\
& + i\omega (-2m_3 \mu + c_3)(k_1 - m_1 \omega^2)(k_2 - m_2 \omega^2)
\end{aligned}$$

It is easily seen that the real part of Equation [17] as thus reduced is the same as Equation [16] for undamped vibration. Hence the frequencies of oscillation in the three damped modes are approximated here by the frequencies of undamped vibration and may be calculated from Equation [16].

To shorten the notation, write now

$$\begin{aligned}
G_1 &= k_1 - m_1 \omega^2, \quad G_2 = k_2 - m_2 \omega^2, \quad G_3 = k_3 - m_3 \omega^2 \\
G_{12} &= k_{12} - m_{12} \omega^2, \quad G_{13} = k_{13} - m_{13} \omega^2, \quad G_{23} = k_{23} - m_{23} \omega^2
\end{aligned}$$

Then it will be found that the imaginary part of Equation [17] divided by  $i\omega$  can be written in its approximated form thus.

$$\begin{aligned}
& -2\mu \left[ m_1(G_2 G_3 - G_{23}^2) + m_2(G_1 G_3 - G_{13}^2) + m_3(G_1 G_2 - G_{12}^2) \right. \\
& + 2m_{12}(G_{13} G_{23} - G_3 G_{12}) + 2m_{13}(G_{12} G_{13} - G_2 G_{13}) \\
& \left. + 2m_{23}(G_{12} G_{13} - G_1 G_{23}) \right] + c_1(G_2 G_3 - G_{23}^2) + c_2(G_1 G_3 - G_{13}^2) \\
& + c_3(G_1 G_2 - G_{12}^2) + (c_{12} + c_{21})(G_{13} G_{23} - G_3 G_{12}) + (c_{13} + c_{31}) \\
& (G_{12} G_{23} - G_2 G_{13}) + (c_{23} + c_{32})(G_{12} G_{13} - G_1 G_{23}) = 0 \quad [18]
\end{aligned}$$

After inserting in the G's the proper value of  $\omega^2$  for any one of the damped modes, this equation is easily solved for an approximate value of the damping constant  $\mu$  for that mode.

### 3. HARMONIC FORCED VIBRATIONS

Assume

$$\begin{aligned} P &= p \cos \omega_0 t + p' \sin \omega_0 t, & Q &= q \cos \omega_0 t + q' \sin \omega_0 t \\ R &= r \cos \omega_0 t + r' \sin \omega_0 t \end{aligned}$$

where  $p, p', q, q', r, r'$  are any six real amplitudes and  $\omega_0$  is any positive real number. For the resulting steady vibration write

$$\begin{aligned} x &= a_1 \cos \omega_0 t + a'_1 \sin \omega_0 t, & y &= a_2 \cos \omega_0 t + a'_2 \sin \omega_0 t \\ z &= a_3 \cos \omega_0 t + a'_3 \sin \omega_0 t \end{aligned}$$

$a_1 \dots a'_3$  being six real numbers.

In any particular motion, by a proper choice of the origin for  $t$ , any chosen one of the six variables  $P, Q, R, x, y, z$  can be made to vibrate in proportion to  $\cos \omega_0 t$ , or to  $\sin \omega_0 t$ . Thus any one of the twelve amplitudes  $p, p', \dots a_3, a'_3$  can be assumed to be zero without altering the motion that is represented.

Substitution in Equations [15a, b, c] and separation of sine and cosine terms gives six equations. To shorten the notation, write:

$$\begin{aligned} F_1 &= k_1 - m_1 \omega_0^2 & F_2 &= k_2 - m_2 \omega_0^2 & F_3 &= k_3 - m_3 \omega_0^2 \\ F_{12} &= k_{12} - m_{12} \omega_0^2 & F_{13} &= k_{13} - m_{13} \omega_0^2 & F_{23} &= k_{23} - m_{23} \omega_0^2 \end{aligned}$$

Then the six equations read:

$$F_1 a_1 + c_1 \omega_0 a_1' + F_{12} a_2 + c_{12} \omega_0 a_2' + F_{13} a_3 + c_{13} \omega_0 a_3' = p \quad [19a]$$

$$-c_1 \omega_0 a_1 + F_1 a_1' - c_{12} \omega_0 a_2 + F_{12} a_2' - c_{13} \omega_0 a_3 + F_{13} a_3' = p' \quad [19b]$$

$$F_{12} a_1 + c_{21} \omega_0 a_1' + F_2 a_2 + c_2 \omega_0 a_2' + F_{23} a_3 + c_{23} \omega_0 a_3' = q \quad [19c]$$

$$-c_{21} \omega_0 a_1 + F_{12} a_1' - c_2 \omega_0 a_2 + F_2 a_2' - c_{23} \omega_0 a_3 + F_{23} a_3' = q' \quad [19d]$$

$$F_{13} a_1 + c_{31} \omega_0 a_1' + F_{23} a_2 + c_{32} \omega_0 a_2' + F_3 a_3 + c_3 \omega_0 a_3' = r \quad [19e]$$

$$-c_{31} \omega_0 a_1 + F_{13} a_1' - c_{32} \omega_0 a_2 + F_{23} a_2' - c_3 \omega_0 a_3 + F_3 a_3' = r' \quad [19f]$$

In general any six of the twelve amplitudes  $a_1 \dots r'$  can be assigned arbitrarily and the equations then fix the values of the other six.

#### 4. EXPERIMENTAL DETERMINATION OF

$$c_1, c_2, c_3, c_{12}, c_{13}, c_{21}, c_{23}, c_{31}, c_{32}$$

The methods described for a two-dimensional system can be extended to three dimensions. Determination of the nine damping constants from general bumping observations, however, will not be discussed here because it appears to involve very complicated observations and calculations.

A feasible alternative might be to lock each of the three coordinates in turn so as to hold it at zero. The given three-dimensional system could thus be studied as a combination of three two-dimensional systems and the methods already described for such systems would be available.

Of three-dimensional motions, only forced harmonic motions will be considered here and only the simplest use of these. In such motions,  $x$ ,  $y$ , and  $z$  are equal respectively to  $-\omega_0^2 x$ ,  $-\omega_0^2 y$ , and  $-\omega_0^2 z$  so that either  $x$ ,  $y$ ,

or  $z$  may be measured.

Apply  $P, Q, R$  in any convenient ratio but all in the same phase.

Assume  $p' = q' = r' = 0$ . Read the resulting three inphase amplitudes  $a_1, a_2, a_3$  and the three quadrature amplitudes  $a'_1, a'_2, a'_3$ , the latter being relatively small. Repeat with different ratios of  $P, Q, R$ , distinguishing the  $a$ 's thus obtained by a bar, and then with a third set of ratios, marking the  $a$ 's with a double bar. A possible choice would be to use only  $P$  the first time, only  $Q$  the second time, and only  $R$  the third time.

Substitution of the first set of observed  $a$ 's in Equations [19a,d,f], then the second set of  $a$ 's, and finally the third set gives three groups of three equations each for the determination of the nine  $c$ 's. Since in all cases  $p' = q' = r' = 0$ , the equations may be slightly rearranged to read as follows:

$$a_1 c_{11} + a_2 c_{12} + a_3 c_{13} = \frac{1}{\omega_0} (F_{11} a_1 + F_{12} a_2 + F_{13} a_3) \quad [20]$$

$$\bar{a}_1 c_{11} + \bar{a}_2 c_{12} + \bar{a}_3 c_{13} = \frac{1}{\omega_0} (F_{11} \bar{a}_1 + F_{12} \bar{a}_2 + F_{13} \bar{a}_3)$$

$$\bar{\bar{a}}_1 c_{11} + \bar{\bar{a}}_2 c_{12} + \bar{\bar{a}}_3 c_{13} = \frac{1}{\omega_0} (F_{11} \bar{\bar{a}}_1 + F_{12} \bar{\bar{a}}_2 + F_{13} \bar{\bar{a}}_3)$$

$$a_1 c_{21} + a_2 c_{22} + a_3 c_{23} = \frac{1}{\omega_0} (F_{21} a_1 + F_{22} a_2 + F_{23} a_3)$$

$$\bar{a}_1 c_{21} + \bar{a}_2 c_{22} + \bar{a}_3 c_{23} = \frac{1}{\omega_0} (F_{21} \bar{a}_1 + F_{22} \bar{a}_2 + F_{23} \bar{a}_3)$$

$$\bar{\bar{a}}_1 c_{21} + \bar{\bar{a}}_2 c_{22} + \bar{\bar{a}}_3 c_{23} = \frac{1}{\omega_0} (F_{21} \bar{\bar{a}}_1 + F_{22} \bar{\bar{a}}_2 + F_{23} \bar{\bar{a}}_3)$$

$$a_1 c_{31} + a_2 c_{32} + a_3 c_{33} = \frac{1}{\omega_0} (F_{31} a_1 + F_{32} a_2 + F_{33} a_3)$$

$$\bar{a}_1 c_{31} + \bar{a}_2 c_{32} + \bar{a}_3 c_3 = \frac{1}{\omega_0} (F_{13} \bar{a}'_1 + F_{23} \bar{a}'_2 + F_3 \bar{a}'_3)$$

$$\bar{a}_1 c_{31} + \bar{a}_2 c_{32} + \bar{a}_3 c_3 = \frac{1}{\omega_0} (F_{13} \bar{a}'_1 + F_{23} \bar{a}'_2 + F_3 \bar{a}'_3)$$

Assuming that the six constants  $F_1, F_2, F_3, F_{12}, F_{13}, F_{23}$  have been calculated from the constants of the system and the chosen value of  $\omega_0$ , the first three of Equations [20] can be solved for  $c_1, c_{12}, c_{13}$ , the middle three for  $c_2, c_{21}, c_{23}$ , and the last three for  $c_3, c_{31}, c_{32}$ .

The computation can be shortened by observing differently. Using chosen  $p$  and  $q$ , adjust  $r$  ( $p', q', r'$  being all zero) so that  $a_3 = 0$ . Read  $a_1, a_2, a'_1, a'_2, a'_3$ . Repeat with a different pair of values of  $p$  and  $q$ , making  $\bar{a}_3 = 0$ . Read  $\bar{a}_1, \bar{a}_2, \bar{a}'_1, \bar{a}'_2, \bar{a}'_3$ . Then the first two of Equations [20] are easily solved for  $c_1$  and  $c_{12}$ , the fourth and fifth for  $c_2$  and  $c_{21}$ , and the seventh and eighth for  $c_{31}$  and  $c_{32}$ .

Repeat using two pairs of values of  $p$  and  $r$  and adjusting  $q$  each time so that  $a_2 = \bar{a}_2 = 0$ . Read  $a_1, a_3, a'_1, a'_2, a'_3$  and  $\bar{a}_1, \bar{a}_3, \bar{a}'_1, \bar{a}'_2, \bar{a}'_3$ . Then the same three pairs out of Equations [20] yield  $c_1$  and  $c_{13}, c_{21}$  and  $c_{23}$ , and  $c_3$  and  $c_{31}$ .

All of the  $c$ 's have thus been obtained, with duplicate values of  $c_1, c_{21}$  and  $c_{31}$ . Other combinations of  $p, q, r$  may be used in a similar way.

It will be noted that neither procedure requires actual measurements of  $P, Q$ , or  $R$ .

## V. VARIABLE DAMPING

In practice there seems to be a tendency for high-frequency vibrations to die out more rapidly than low-frequency vibrations. Such differences may result in many ways from the characteristics of the systems. It is worth noting, however, that

- (1) A simple increase of scale is likely to lower the damping rate.
- (2) The damping rate of a high-frequency mode of vibration can be less than that of a low-frequency mode of the same system.

### 1. CHANGE OF SCALE

As a simple example, consider a mass on a spring subject to linear damping, its equation of motion being

$$m\ddot{x} + c\dot{x} + kx = 0$$

In a damped vibration

$$x = Ae^{-\mu t} \sin \omega t$$

where  $\mu = c/(2m)$  and  $\omega^2 = (k/m) - \mu^2$

Now let all linear dimensions be changed in any ratio  $\lambda$  without change of material. Then\*  $m \propto \lambda^3$ ,  $k \propto \lambda^2$ . What happens to  $c$ ? At given  $\dot{x}$ , water resistance will tend to vary in proportion to the surface wetted; hence  $c \propto \lambda^2$ . For simplicity, suppose  $\mu^2$  may be dropped in comparison with  $k/m$ . Then, approximately,

\* In a change of scale including change of both cross section and length of spring,  $k \propto \frac{\lambda^2}{\lambda} = \lambda$ . For a mass on a spring, when all dimensions change,  $m \propto \lambda^3$ ,  $k \propto \lambda$ ,  $\omega \propto \frac{1}{\lambda}$ . If only the length of the spring does not change,  $m \propto \lambda^3$ ,  $k \propto \lambda^2$ ,  $\omega \propto \frac{1}{\sqrt{\lambda}}$

$$\omega \propto 1/\sqrt{\lambda} ; \mu \propto 1/\lambda ; \therefore \mu \propto \omega^2$$

Or, if  $c$  does not change when  $\lambda \neq 1$ , then approximately when  $\mu$  is small

$$\omega \propto 1/\sqrt{\lambda} ; \mu \propto 1/\lambda^3 ; \therefore \mu \propto \omega^6$$

In both cases  $\mu$  and  $\omega$  both increase if  $\lambda < 1$  and decrease if  $\lambda > 1$ , thus varying "in the same direction."

## 2. CONTRARY MODES FOR A GIVEN SYSTEM

Since higher frequency tends to mean higher velocities at a given amplitude, it might reasonably be guessed that the damping will be greater in modes of higher frequency. This is not necessarily the case, however, because the components of displacement are in different ratios in different modes and some components may be damped more heavily than others.

As a simple example, suppose

$$m_1 \ddot{x} + k_1 x + k_{12} y + c_1 \dot{x} = 0 \quad m_2 \ddot{y} + k_2 y + k_{12} x + c_2 \dot{y} = 0$$

where  $k_2/m_2 \gg k_1/m_1$  but  $c_2/m_2 \ll c_1/m_1$ .

If  $k_{12} = 0$  and  $c_1 = c_2 = 0$ , then in one mode,  $x$  vibrates with  $y = 0$ ; in the other,  $y$  vibrates at much higher frequency with  $x = 0$ . If  $k_{12} = 0$  but  $c_1$  and  $c_2$  are merely small, then the two frequencies are little altered by  $c_1$  and  $c_2$ , and the damping will be much less for the second or  $y$  vibration than for the first or  $x$  vibration.

Thus higher frequency is accompanied here by lower damping. This conclusion will not be altered if  $k_{12}$  is merely kept small but not zero, so that  $y$  vibrates a little in the first mode and  $x$  vibrates a little in the second mode.

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Table 1  
Summary of Results  
One Dimensional Damped Vibrations  
 $m\ddot{x} + c\dot{x} + kx = P(t)$

CASE	SOLUTIONS	REMARKS	EXPERIMENTAL DETERMINATION OF DAMPING
(1) DAMPED FREE VIBRATIONS $P = 0$	When $c^2 < 4mk$ (less than critical damping), $x = e^{-\mu t} (a \cos \omega_d t + b \sin \omega_d t)$ where $\mu = \frac{c}{2m}$ , $\omega_d^2 = \frac{1}{m} \left( k - \frac{c^2}{4} \right)$ $c^2 = 4mk$ (critical damping), $x = (a + bte^{-\mu t})$ where $\mu = \frac{c}{2m}$ $c^2 > 4mk$ (greater than critical damping), $x = ae^{-\mu_1 t} + be^{-\mu_2 t}$ where $\mu_{1,2} = \frac{1}{2m} (c \pm \sqrt{c^2 - 4mk})$ $a, b$ arbitrary	(1) Two independent modes of damped free vibration can occur. Their amplitudes can be chosen to make $x$ and $\dot{x}$ agree with any assumed initial values. (2) The free vibrations are oscillatory provided the damping constant $c$ is not too large. Small $c$ has negligible effect on the oscillatory frequency.	(1) Obtain $\mu$ from curve of $x$ vs $t$ (when $\mu \neq 0$ ). Then $\alpha = 2\mu$ .
(2) HARMONIC FORCED VIBRATIONS $P(t) = p \cos \omega_0 t$	$x = a \cos \omega_0 t + b \sin \omega_0 t$ $p^2 = \left[ \frac{a^2}{k} - \frac{m\omega_0^2}{k} + c^2 \omega_0^2 \right]$ $\frac{a}{b} = \frac{k - m\omega_0^2}{c\omega_0}$ $p, \omega_0$ arbitrary	(1) Introduces a component of $x$ in time quadrature relative to $P$ . When $\omega_0 \gg \omega_d$ , $a \approx 1$ and the vibration is in time quadrature only relative to $P$ . (2) $ x _{\max}$ for given $p$ occurs when $\omega_0^2 = \frac{k}{m} - \frac{c^2}{2m^2}$	(1) Observe ratio of quadrature $\epsilon$ and in phase components of $x$ relative to the phase of $P$ . Then $\epsilon = \frac{1}{\omega_0} \left( k - m\omega_0^2 \right) \frac{b}{a}$ .
TWO DIMENSIONAL DAMPED VIBRATIONS			
(1) UNDAMPED FREE VIBRATIONS $P = Q = 0$ $c_1 = c_2 = c_{12} = c_{21} = 0$	When $x$ and $y$ vibrate together in proportion to $\cos \omega t$ (i.e., $x/y \propto \cos \omega t$ ) $\frac{y}{x} = \frac{k_{12} - m_{12}\omega^2}{k_2 - m_2\omega^2} = \frac{k_1 - m_1\omega^2}{k_{12} - m_{12}\omega^2}$ Where the $\omega$ 's are determined from $(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - (k_{12} - m_{12}\omega^2)^2 = (m_1m_2 - m_{12}^2)\omega^4$ $-(m_1k_2 + m_2k_1 - 2m_{12}k_{12})\omega^2 + (k_1k_2 - k_{12}^2) = 0$	(1) When $\omega = \sqrt{\frac{k_1}{m_1}}$ and either $m_{12} = m_{12} = 0$ or $m_{12} = m_{12}$ , then $x$ vibrates and $y = 0$ . When $\omega = \sqrt{\frac{k_2}{m_2}}$ and either $m_{12} = m_{12} = 0$ or $m_{12} = m_{12}$ , then $y$ vibrates and $x = 0$ . (2) When $x$ and $y$ vibrate together then in general two independent modes of vibration occur with $y$ and $x$ in fixed ratio depending on the mode. (e) If $k_{12}^2 = k_1k_2$ , then one root of the	EXPERIMENTAL DETERMINATION OF DAMPING





(2) DAMPED FREE VIBRATIONS $P = Q = 0$	$x = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$ , $y = b_1 e^{\lambda_1 t} + b_2 e^{\lambda_2 t}$ where $\lambda$ is determined from $c_1 \lambda^4 + c_2 \lambda^3 + c_3 \lambda^2 + c_4 \lambda + c_5 = 0$ $a, b, \lambda$ non-zero constants real or complex (negative real or complex) $c_1 \neq 0$ (if $m, k, c$ are positive or zero (see equation 10)) For $\lambda$ complex ( $m = \mu + i\nu$ ) $\mu$ and $\nu$ are positive and determined by successive approximation from $c_1 \mu^4 - (c_2 - 3c_3 \mu + 6c_4 \mu^2) \mu^2 + c_5 - c_1 \mu^2 + c_2 \mu^2 - c_3 \mu^3 + c_4 \mu^4 = 0$ $c_1 - c_3 \mu^2 - (2c_2 - 4c_4 \mu^2) \mu + 3c_5 \mu^2 - 4c_3 \mu^3 = 0$ For small damping $\mu = \frac{1}{2} \left( \frac{c_2 - c_3 \mu^2}{c_1 - c_3 \mu^2} \right)$ More accurate solutions are determined by successive approximation from the two equations immediately above. and $w$ is determined from $c_4 w^4 - c_5 w^2 + c_6 = 0$ In general $\frac{\lambda}{2} = \frac{m_1 \lambda^2 + k_1 + c_1 \lambda}{m_{12} \lambda^2 + k_{12} + c_{12} \lambda} \pm \frac{m_1 \lambda^2 + k_1 + c_1 \lambda}{m_2 \lambda^2 + k_2 + c_2 \lambda}$	second root for $w^2$ occurs at a value greater than these quantities.	(1) When $m_{12} = k_{12} = c_{12} = 0$ then $x$ vibrates and $y = 0$ . When $m_{12} = k_{12} = c_{12} = 0$ then $y$ vibrates and $x = 0$ . (2) Since $\frac{m_{12}}{m_1} = \frac{b}{a}$ is complex there is generally a phase difference between $x$ and $y$ as functions of time. (3) The general solution is the sum of four real expressions representing four independent damped oscillations whose amplitudes can be chosen to agree with any assumed initial (or other) time values of $x, \dot{x}, y, \dot{y}$ . (4) For small damping $\mu$ is small and the oscillation frequencies are the $\omega$ for the undamped case.
(3) HARMONIC FORCED VIBRATIONS $P = p \cos w_0 t + p' \sin w_0 t$ $Q = q \cos w_0 t + q' \sin w_0 t$	$x = a_1 \cos w_0 t + a_2 \sin w_0 t$ , $y = b_1 \cos w_0 t + b_2 \sin w_0 t$ $p, p', q, q', a_1, a_2, b_1, b_2$ eight real numbers, four arbitrary, one can be zero. Determination of amplitudes $a_1, a_2, b_1, b_2$ caused by applied forces $p, p', q, q'$ is made by solving $(k_1 - m_1 w_0^2) a_1 + c_1 w_0 a_1 + (k_{12} - m_{12} w_0^2) a_2 + c_{12} w_0 a_2 = p$ $-c_1 w_0 a_1 + (k_1 - m_1 w_0^2) a_1 - c_{12} w_0 a_2 + (k_{12} - m_{12} w_0^2) a_2 = p'$ $(k_{12} - m_{12} w_0^2) a_1 + c_{12} w_0 a_1 + (k_2 - m_2 w_0^2) a_2 + c_2 w_0 a_2 = q$ $-c_{12} w_0 a_1 + (k_{12} - m_{12} w_0^2) a_1 - c_2 w_0 a_2 + (k_2 - m_2 w_0^2) a_2 = q'$ For these equations the determinant of the coefficients $a_1, a_2, b_1, b_2$ is $\Delta = \begin{vmatrix} (k_1 - m_1 w_0^2) & c_1 w_0 & (k_{12} - m_{12} w_0^2) & c_{12} w_0 \\ -c_1 w_0 & (k_1 - m_1 w_0^2) & (k_{12} - m_{12} w_0^2) & c_{12} w_0 \\ (k_{12} - m_{12} w_0^2) & c_{12} w_0 & (k_2 - m_2 w_0^2) & c_2 w_0 \\ -c_{12} w_0 & (k_{12} - m_{12} w_0^2) & -c_2 w_0 & (k_2 - m_2 w_0^2) \end{vmatrix}$	(1) If there is no damping, then when $w_0$ equals the value of $w$ for either of the frequencies of undamped vibration $\Delta = 0$ and the amplitude of the forced vibration becomes large (i.e. resonance occurs). (2) If $c_1, c_2, c_{12}, c_{21}$ are all small $\Delta = 0$ at two slightly modified frequencies that differ also slightly from the frequencies of damped free oscillation. As $w_0$ approaches either of these frequencies at which $\Delta = 0$ while $p, p', q, q'$ remain fixed, the amplitude of the forced vibration becomes large (i.e. resonance occurs).	(1) Apply $P = p \cos w_0 t$ only (i.e. $p' = 0$ ) and adjust amplitude and phase of $Q$ (as discussed in text) so that $a_2 = a_2' = 0$ (i.e. $y = 0$ ). With $w_0 > \sqrt{\frac{k_1}{m_1}}$ observe $\frac{a_1}{m_1}$ . Then $c_1 = \frac{a_1}{w_0^2} (k_1 - m_1 w_0^2)$ . (2) Similarly, apply $Q = q \cos w_0 t$ only (i.e. $q' = 0$ ) and make $x = 0$ by adjusting $p$ and $p'$ so that $a_1 = a_1' = 0$ . With $w_0 > \sqrt{\frac{k_2}{m_2}}$ observe $\frac{a_2}{m_2}$ . Then $c_2 = \frac{a_2}{w_0^2} (k_2 - m_2 w_0^2)$ . (3) Apply $P = p \cos w_0 t$ only (i.e. $p' = 0$ ). Adjust $Q$ (as discussed in text) so that $a_1' = a_2' = 0$ . Read $a_1/a_2$ . Then $c_{12} = -\frac{a_1}{a_2} c_1$ . (4) Similarly, apply $Q = q \cos w_0 t$ only (i.e. $q' = 0$ ). Adjust $P$ so that $a_1' = a_2' = 0$ . Read $\frac{a_2}{a_1}$ . Then $c_{21} = -\frac{a_2}{a_1} c_2$ . (5) Apply $P = p \cos w_0 t$ only (i.e. $p' = q' = 0$ ). Read $a_1, a_2, a_1', a_2'$ . Repeat with different ratio $\frac{Q}{P} = \frac{q}{p}$ and read corresponding (barred) values $\bar{a}_1, \bar{a}_2, \bar{a}_1', \bar{a}_2'$ . Substitute each set of $a$ 's in turn in second and fourth of the set of four equations given in "solution" in which $p' = q' = 0$ . Solve resulting two pairs of equations for $c_1, c_{12}$ and $c_2, c_{21}$ .

Table 1 - Summary of Results (Continued)

Three Dimensional Damped Vibrations

$$m_1 \ddot{x} + k_1 x + m_{12} \ddot{y} + k_{12} x + m_{13} \ddot{z} + k_{13} x + c_1 \dot{x} + c_{12} \dot{y} + c_{13} \dot{z} = P(t)$$

$$m_{12} \ddot{x} + k_{12} x + m_2 \ddot{y} + k_2 y + m_{23} \ddot{z} + k_{23} y + c_{21} \dot{x} + c_2 \dot{y} + c_{23} \dot{z} = Q(t)$$

$$m_{13} \ddot{x} + k_{13} x + m_{23} \ddot{y} + k_{23} y + m_3 \ddot{z} + k_3 z + c_{31} \dot{x} + c_{32} \dot{y} + c_3 \dot{z} = R(t)$$

CASE	SOLUTIONS	REMARKS	EXPERIMENTAL DETERMINATION OF DAMPING
(1) UNDAMPED FREE VIBRATIONS P = Q = R = 0 $c_1 = c_2 = c_{12} = c_{21} = c_{13} = c_{31} = c_{23} = c_{32} = 0$	$x = a_1 \cos \omega t, y = a_2 \cos \omega t, z = a_3 \cos \omega t; a_1, a_2, a_3 \text{ real}$ Ratios $\frac{x}{y} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \frac{a_3}{a_1}$ or their inverses are found by solving In any one of the three modes: $(k_1 - m_1 \omega^2)a_1 + (k_{12} - m_{12} \omega^2)a_2 + (k_{13} - m_{13} \omega^2)a_3 = 0$ $(k_{12} - m_{12} \omega^2)a_1 + (k_2 - m_2 \omega^2)a_2 + (k_{23} - m_{23} \omega^2)a_3 = 0$ $(k_{13} - m_{13} \omega^2)a_1 + (k_{23} - m_{23} \omega^2)a_2 + (k_3 - m_3 \omega^2)a_3 = 0$ where the $\omega$ 's for these modes are determined from the cubic equation in $\omega^2$ $\Delta = (k_1 - m_1 \omega^2)(k_2 - m_2 \omega^2)(k_3 - m_3 \omega^2) + (k_{12} - m_{12} \omega^2)(k_{13} - m_{13} \omega^2)(k_{23} - m_{23} \omega^2) - (k_1 - m_1 \omega^2)(k_{23} - m_{23} \omega^2)^2 - (k_2 - m_2 \omega^2)(k_{13} - m_{13} \omega^2)^2 - (k_3 - m_3 \omega^2)(k_{12} - m_{12} \omega^2)^2 = 0$	(1) When $x, y$ and $z$ vibrate together then in general three independent modes of vibration occur with $\omega$ 's in fixed ratio to each other depending upon the mode	
(2) DAMPED FREE VIBRATION P = Q = R = 0	$x = a_1 e^{\lambda t}, y = a_2 e^{\lambda t}, z = a_3 e^{\lambda t}; a_1, a_2, a_3, \lambda \text{ generally complex}$ Ratios $x/y/z$ are found by solving in any one of the three damped modes: $(k_1 + m_1 \lambda^2 + c_1 \lambda)a_1 + (k_{12} + m_{12} \lambda^2 + c_{12} \lambda)a_2 + (k_{13} + m_{13} \lambda^2 + c_{13} \lambda)a_3 = 0$ $(k_{12} + m_{12} \lambda^2 + c_{21} \lambda)a_1 + (k_2 + m_2 \lambda^2 + c_2 \lambda)a_2 + (k_{23} + m_{23} \lambda^2 + c_{23} \lambda)a_3 = 0$ $(k_{13} + m_{13} \lambda^2 + c_{31} \lambda)a_1 + (k_{23} + m_{23} \lambda^2 + c_{32} \lambda)a_2 + (k_3 + m_3 \lambda^2 + c_3 \lambda)a_3 = 0$ Where the $\lambda$ 's for these modes are determined from the sixth degree	(1) The sixth degree equation in $\lambda, m, y$ have roots or all roots of the $\lambda$ 's are large enough. If the $\lambda$ 's are not too large there will be six complex roots $\lambda_k, k = 1, 2, 3$ . (2) For small $\lambda$ 's and therefore small $\lambda$ 's the frequencies of oscillation in the three damped modes are approximated by the frequencies of undamped vi-	(1) Methods of measurement similar to those for the two-dimensional case; elaborate procedure necessary to isolate the nine $c$ 's in turn.

$$\omega_{\lambda} = \omega_{\lambda}^0, \lambda = 1, 2, 3.$$

(2) For small  $\epsilon$ 's and therefore small  $\mu$ 's the frequency of oscillation in the three damped modes are approximated by the frequencies of undamped vibrations obtained from the cubic equation in  $\omega^2$ .

$$\begin{aligned} & (\lambda_{12}^2 + m_{12}^2) \lambda^2 + 2m_{12} \lambda + c_{12}^2 = 0 \\ & (\lambda_{23}^2 + m_{23}^2) \lambda^2 + 2m_{23} \lambda + c_{23}^2 = 0 \\ & (\lambda_{13}^2 + m_{13}^2) \lambda^2 + 2m_{13} \lambda + c_{13}^2 = 0 \end{aligned}$$

Where the  $\lambda$ 's for these modes are determined from the sixth degree equation in  $\lambda$

$$\begin{aligned} & (\lambda_1^2 + m_1^2) \lambda^2 + c_1^2 \lambda + c_2^2 \lambda (\lambda_2^2 + m_2^2) + c_3^2 \lambda (\lambda_3^2 + m_3^2) \\ & + (\lambda_{12}^2 + m_{12}^2) \lambda^2 + c_{12}^2 \lambda + c_{23}^2 \lambda (\lambda_{13}^2 + m_{13}^2) + c_{31}^2 \lambda \\ & + (\lambda_{12}^2 + m_{12}^2) \lambda^2 + c_{12}^2 \lambda + c_{23}^2 \lambda (\lambda_{13}^2 + m_{13}^2) + c_{31}^2 \lambda \\ & - (\lambda_1^2 + m_1^2) \lambda^2 + c_1^2 \lambda (\lambda_{23}^2 + m_{23}^2) + c_2^2 \lambda (\lambda_{13}^2 + m_{13}^2) + c_{32}^2 \lambda \\ & - (\lambda_2^2 + m_2^2) \lambda^2 + c_2^2 \lambda (\lambda_{13}^2 + m_{13}^2) + c_3^2 \lambda (\lambda_{12}^2 + m_{12}^2) + c_{31}^2 \lambda \\ & - (\lambda_3^2 + m_3^2) \lambda^2 + c_3^2 \lambda (\lambda_{12}^2 + m_{12}^2) + c_{12}^2 \lambda (\lambda_{13}^2 + m_{13}^2) + c_{21}^2 \lambda = 0 \end{aligned}$$

For each mode or value of  $\omega^2$  found from the cubic equation in  $\omega^2$  for undamped free vibrations (an approximation for small damping) a corresponding  $\mu$  is obtained from

$$\begin{aligned} & -2\mu \left[ m_1(G_1G_2 - G_2G_3) + m_2(G_1G_3 - G_2G_1) + m_3(G_2G_1 - G_1G_2) \right. \\ & \quad \left. + 2m_{12}(G_1G_{23} - G_2G_{12}) + 2m_{13}(G_1G_{23} - G_2G_{13}) \right. \\ & \quad \left. + 2m_{23}(G_1G_{13} - G_2G_{23}) \right] \\ & + c_1(G_2G_3 - G_2G_1) + c_2(G_1G_3 - G_1G_2) + c_3(G_1G_2 - G_2G_1) \\ & + (c_{12} + c_{21})(G_1G_{23} - G_2G_{12}) + (c_{13} + c_{31})(G_1G_{23} - G_2G_{13}) \\ & + (c_{23} + c_{32})(G_1G_{13} - G_2G_{23}) = 0 \end{aligned}$$

Where

$$\begin{aligned} G_1 &= k_1 - m_1\omega^2, \quad G_2 = k_2 - m_2\omega^2, \quad G_3 = k_3 - m_3\omega^2 \\ G_{12} &= k_{12} - m_{12}\omega^2, \quad G_{13} = k_{13} - m_{13}\omega^2, \quad G_{23} = k_{23} - m_{23}\omega^2 \end{aligned}$$

### (3) HARMONIC FORCED VIBRATIONS

$$\begin{aligned} P &= p \cos \omega_0 t + p' \sin \omega_0 t \\ Q &= q \cos \omega_0 t + q' \sin \omega_0 t \\ R &= r \cos \omega_0 t + r' \sin \omega_0 t \end{aligned}$$

$$x = a_1 \cos \omega_0 t + a_1' \sin \omega_0 t, \quad y = a_2 \cos \omega_0 t + a_2' \sin \omega_0 t$$

$P, p'; Q, q'; R, r'$ ,  $a_1, a_1'; a_2, a_2'$ ,  $a_3, a_3'$  twelve real numbers, six arbitrary, one can be zero. In general, any six of these numbers can be assigned arbitrarily and the remaining six are found by solving

$$\begin{aligned} & F_{11}a_1 + c_{11}\omega_0^2 a_1 + F_{12}a_2 + c_{12}\omega_0^2 a_2 + F_{13}a_3 + c_{13}\omega_0^2 a_3 = p \\ & -c_{12}\omega_0^2 a_1 + F_{11}a_1 - c_{12}\omega_0^2 a_2 + F_{12}a_2 - c_{13}\omega_0^2 a_3 + F_{13}a_3 = p' \\ & F_{12}a_1 + c_{21}\omega_0^2 a_1 + F_{22}a_2 + c_{22}\omega_0^2 a_2 + F_{23}a_3 + c_{23}\omega_0^2 a_3 = q \\ & -c_{12}\omega_0^2 a_1 + F_{12}a_2 - c_{22}\omega_0^2 a_2 + F_{22}a_2 - c_{23}\omega_0^2 a_3 + F_{23}a_3 = q' \\ & F_{13}a_1 + c_{31}\omega_0^2 a_1 + F_{23}a_2 + c_{32}\omega_0^2 a_2 + F_{33}a_3 + c_{33}\omega_0^2 a_3 = r \\ & -c_{31}\omega_0^2 a_1 + F_{31}a_1 - c_{32}\omega_0^2 a_2 + F_{23}a_3 - c_{33}\omega_0^2 a_3 + F_{33}a_3 = r' \end{aligned}$$

where

$$\begin{aligned} F_{11} &= k_1 - m_1\omega_0^2, \quad F_{22} = k_2 - m_2\omega_0^2, \quad F_{33} = k_3 - m_3\omega_0^2 \\ F_{12} &= k_{12} - m_{12}\omega_0^2, \quad F_{13} = k_{13} - m_{13}\omega_0^2, \quad F_{23} = k_{23} - m_{23}\omega_0^2 \end{aligned}$$

(1) Apply  $P, Q, R$  in any ratio but in same phase. Assume  $p' = q' = r' = 0$ . Read  $a_1, a_2, a_3, a_1', a_2', a_3'$ . Repeat with different ratios of  $P, Q, R$ , distinguishing  $a$ 's thus obtained by a bar, and then with a third set of ratios, making the  $a$ 's with a double bar. (see suggestions for choosing  $P, Q, R$  in text). Substitute each set of  $a$ 's in turn in second, fourth and sixth equations of the set of six equations given in "solution" in which  $p' = q' = r' = 0$ . Solve resulting three groups of three equations for the nine

$c$ 's (a rearrangement of the nine equations convenient for computation, is given in text (Equation (20)). With this rearrangement each of the three groups of three equations is solved separately, the first group yielding  $c_1, c_{12}, c_{13}$ , the second group yielding  $c_2, c_{21}, c_{23}$  and the last group  $c_3, c_{31}, c_{32}$ .)

(2) Computation is further shortened by observing differently.

Choose  $p$  and  $q$  and adjust  $r$  ( $p' = q' = r' = 0$ ) so that  $a_3 = 0$ . Read  $a_1, a_2, a_1', a_2', a_3'$ . Repeat with different pair of values of  $p$  and  $q$  making  $a_3 = 0$ . Read  $a_1, a_2, a_1', a_2', a_3'$ . Solve first two of rearranged equations (Equation (20) in text) for  $c_1$  and  $c_2$ .

For each mode or value of  $\omega^2$  found from the cubic equation in  $\omega^2$  for undamped free vibrations (an approximation for small damping) a corresponding  $\mu$  is obtained from

$$-2\mu \left[ m_1(G_1G_2 - G_2G_3) + m_2(G_1G_3 - G_2G_1) + m_3(G_1G_2 - G_2G_1) \right. \\ \left. + 2m_1(G_1G_2 - G_2G_1) + 2m_2(G_1G_3 - G_2G_1) + 2m_3(G_1G_2 - G_2G_1) \right] \\ + c_1(G_1G_2 - G_2G_3) + c_2(G_1G_3 - G_2G_1) + c_3(G_1G_2 - G_2G_1) \\ + (c_{12} + c_{21})(G_1G_2 - G_2G_1) + (c_{13} + c_{31})(G_1G_3 - G_2G_1) \\ + (c_{23} + c_{32})(G_1G_2 - G_2G_1) = 0$$

Where

$$G_1 = k_1 - m_1\omega^2, \quad G_2 = k_2 - m_2\omega^2, \quad G_3 = k_3 - m_3\omega^2 \\ G_{12} = k_{12} - m_{12}\omega^2, \quad G_{13} = k_{13} - m_{13}\omega^2, \quad G_{23} = k_{23} - m_{23}\omega^2$$

### (3) HARMONIC FORCED VIBRATION

$$P = p \cos \omega t + p' \sin \omega t \\ Q = q \cos \omega t + q' \sin \omega t \\ R = r \cos \omega t + r' \sin \omega t$$

$x = a_1 \cos \omega t + a_1' \sin \omega t, \quad y = a_2 \cos \omega t + a_2' \sin \omega t$   
 $z = a_3 \cos \omega t + a_3' \sin \omega t$   
P, P', Q, Q', R, R',  $a_1, a_1', a_2, a_2', a_3, a_3'$  twelve real numbers, six arbitrary, one can be zero. In general, any six of these numbers can be assigned arbitrarily and the remaining six are found by solving

$$F_{11}a_1 + c_{11}\omega a_1' + F_{12}a_2 + c_{12}\omega a_2' + F_{13}a_3 + c_{13}\omega a_3' = p \\ -c_{12}\omega a_2' + F_{11}a_1' - c_{13}\omega a_3' + F_{12}a_2' - c_{13}\omega a_3' + F_{13}a_3' = p' \\ F_{12}a_1 + c_{21}\omega a_1' + F_{22}a_2 + c_{22}\omega a_2' + F_{23}a_3 + c_{23}\omega a_3' = q \\ -c_{12}\omega a_1' + F_{22}a_2' - c_{23}\omega a_3' + F_{23}a_3' = q' \\ F_{13}a_1 + c_{31}\omega a_1' + F_{23}a_2 + c_{32}\omega a_2' + F_{33}a_3 + c_{33}\omega a_3' = r \\ -c_{31}\omega a_1' + F_{33}a_3' - c_{32}\omega a_2' + F_{23}a_2' - c_{33}\omega a_3' + F_{33}a_3' = r' \\ \text{where} \\ F_{11} = k_1 - m_1\omega^2, \quad F_{12} = k_{12} - m_{12}\omega^2, \quad F_{13} = k_{13} - m_{13}\omega^2 \\ F_{22} = k_2 - m_2\omega^2, \quad F_{23} = k_{23} - m_{23}\omega^2 \\ F_{33} = k_3 - m_3\omega^2$$

(1) Apply P, Q, R in any ratio but in same phase. Assume  $p' = q' = r' = 0$ . Read  $a_1, a_2, a_3, a_1', a_2', a_3'$ . Repeat with different ratios of P, Q, R, distinguishing a's thus obtained by a bar, and then with a third set of ratios, making the a's with a double bar, (see suggestions for choosing P, Q, R in text). Substitute each set of a's in turn in second, fourth and sixth equations of the set of six equations given in "solutions" in which  $p' = q' = r' = 0$ . Solve resulting three groups of three equations for the nine

c's (a rearrangement of the nine equations convenient for computation, is given in text (Equation (20)). With this rearrangement each of the three groups of three equations is solved separately, the first group yielding  $c_1, c_{12}, c_{13}$ , the second group yielding  $c_2, c_{21}, c_{23}$  and the last group  $c_3, c_{31}, c_{32}$ .) Computation is further shortened by observing differently.

Choose p and q and adjust r ( $p' = q' = r' = 0$ ) so that  $a_3 = 0$ . Read  $a_1, a_2, a_1', a_2'$ . Repeat with different pair of values of p and q making  $a_3 = 0$ . Read  $a_1, a_2, a_1', a_2'$ . Solve first two of rearranged equations (Equation (20) in text) for  $c_1$  and  $c_{12}$ , the fourth and fifth for  $c_2$  and  $c_{21}$  and the seventh and eighth for  $c_{31}$  and  $c_{32}$ . Repeat choosing two pairs of values of p and r and adjusting q each time so that  $a_2 = 0$ . Then the same three pairs out of the rearranged equations yield  $c_{21}$  and  $c_{23}$ , and  $c_3$  and  $c_{31}$ . All c's thus obtained are duplicate values of  $c_1, c_{21}, c_{31}$ .

NOTE: Neither (1) nor (2) require actual measurements of P, Q or R.

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